Test 2 Numerical Mathematics 2
October, 2021

Duration: 5 quarters of an hour.

In front of the questions one finds the points. The sum of the points plus 1 gives the end mark of this test. Use of a calculator is allowed.

1. Consider the matrix
$$A = \begin{bmatrix} -2 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -4 & 2 & 2 \\ 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 3 & 0 & -4 \end{bmatrix}.$$
   (a) [1.0] Show that A is reducible.
   (b) [1.5] Localize the eigenvalues of A using the Gershgorin theorems and show that all the eigenvalues are in the negative half plane, i.e. $Re(\lambda) < 0$.
2. [1.5] Let A be a real symmetric matrix. Let x, with $||x||_2 = 1$, and θ be a Ritz pair obtained from the Lanczos method. Show that
$$||Ax - \theta x||_2 > \min_{\lambda \in \sigma(A)} |\lambda - \theta|.$$
3. In the table below, 10 successive approximations of an eigenvalue $\lambda^{(i)}$ (i is the iteration number) during the Power iteration are shown.

i	$\lambda^{(i)}$
1	3.85638
2	1.20875
3	1.03569
4	1.00696
5	1.00140
6	1.00028
7	1.00006
8	1.00001
9	1.00000
10	1.00000

   (a) [1.0] To which eigenvalue does this iteration clearly converge and what is the rate of convergence rounded to 2 significant digit?
   (b) [1.5] Actually the matrix occurring in the power iteration is $(A - 7I)^{-1}$ for some A. To which eigenvalue of A is the sequence of the previous part converging. And which eigenvalue of A also plays a role in the convergence rate? (If you were not able to solve the previous part you may take here 2 as the eigenvalue it converged to and the convergence rate 0.25)
4. (a) [1.5] Explain how the Krylov subspace can be used to find approximate eigenvalues and eigenvectors.
   (b) [1.0] Make plausible that for Krylov subspaces it holds that $K^m(A, v) = K^m(A - \alpha I, v)$ for any number α . Explain how this makes a difference for its convergence with respect to that of the Power method.